EFFECTIVELY SEMI-RELATIVISTIC HAMILTONIANS OF NONRELATIVISTIC FORM

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Abstract

We construct effective Hamiltonians which despite their apparently nonrelativistic form incorporate relativistic effects by involving parameters which depend on the relevant momentum. For some potentials the corresponding energy eigenvalues may be determined analytically. Applied to two-particle bound states, it turns out that in this way a nonrelativistic treatment may indeed be able to simulate relativistic effects. Within the framework of hadron spectroscopy, this lucky circumstance may be an explanation for the sometimes extremely good predictions of nonrelativistic potential models even in relativistic regions.

1   Introduction

The fundamental disadvantage inherent to any (semi-) relativistically consistent description of some quantum-theoretic system is obviously brought about by the nonlocality of the "square-root" operator of the relativistically correct kinetic energy, $\sqrt{\vec{p}^2 + m^2}$, entering necessarily in the Hamiltonian H which governs the dynamics of the system under consideration. In contrast to the nonrelativistic limit, obtained from the expansion of the square root up to the lowest $\vec{p}^2$-dependent order, $\sqrt{\vec{p}^2 + m^2} = m + \vec{p}^2/(2\,m) + \ldots$, the presence of the relativistic kinetic-energy operator prevents, in general, a thoroughly analytic discussion; one is forced to rely on some numerical solution of the problem.

This inconvenience may be circumvented—at least in principle—by approximating a given semi-relativistic Hamiltonian H (incorporating, by definition, relativistic kinematics) by the corresponding "effectively semi-relativistic" Hamiltonian, formulated and investigated according to the lines proposed in the present work. These effective Hamiltonians are characterized by their rigorous maintenance of the easier to handle nonrelativistic kinematics while resembling the relativistic formalisms to the utmost possible extent by replacing their intrinsic parameters by effective ones which depend in a well-defined manner on the square of the momentum $\vec{p}$.

In order to be as concrete as possible, we choose to illustrate our route of constructing and evaluating these effectively semi-relativistic Hamiltonians for the particular case of bound states of two particles of spin zero. For simplicity, let us assume that the two constituents of these bound states are of equal mass m; the generalization to different masses is then straightforward. In the framework of a semi-relativistic description all the forces acting between these two particles may be derivable from some coordinate-dependent interaction potential $V(\vec{x})$. Consequently, the semi-relativistic Hamiltonian describing this system in the center-of-momentum frame of its constituents is given by

$$H = 2\sqrt{\vec{p}^2 + m^2} + V(\vec{x}) \quad . \tag{1}$$

The equation of motion resulting from this type of Hamiltonian is usually called "spinless Salpeter equation." As it stands, it represents a standard approximation to the Bethe–Salpeter formalism for bound states within a relativistic quantum field theory. It is derived from the Bethe–Salpeter equation [1]

1. by eliminating—in accordance with the spirit of an instantaneous interaction—any dependence on timelike variables, which leads to the so-called "Salpeter equation" [2], and

2. by neglecting any reference to the spin degrees of freedom of the two involved bound-state constituents and restricting to solutions corresponding exclusively to positive energy.

The outline of this paper is as follows. We introduce in Sect. 2 the effectively semi-relativistic Hamiltonians corresponding to the really semi-relativistic Hamiltonians H of Eq. (1) in their most general form and derive in Sect. 3 for the special case of power-law potentials some sort of "master equation" for that central quantity the knowledge of which enables us to imitate the effects of relativistic kinematics within a formally nonrelativistic framework, namely, the expectation value of the square of the momentum $\vec{p}$. From the consideration of the most important prototypes of interaction potentials in Sect. 4 we are led to conclude, in Sect. 5, that our effective Hamiltonians represent indeed a viable alternative to the original semi-relativistic Hamiltonians (1).

2 Effectively Semi-Relativistic Hamiltonians

The main idea of our way of constructing effectively semi-relativistic Hamiltonians has already been sketched in Refs. [3, 4]. The starting point of this construction is a trivial but nevertheless fundamental inequality. This inequality relates the expectation values, taken with respect to (at this stage) arbitrary Hilbert-space vectors $|\rangle$ normalized to unity, of both the first and second powers of an Hermitian (or, to be more precise, self-adjoint) but otherwise arbitrary operator $\mathcal{O} = \mathcal{O}^\dagger$; it reads

$$|\langle \mathcal{O} \rangle| \leq \sqrt{\langle \mathcal{O}^2 \rangle} \quad .$$

Application of the above inequality to the relativistic kinetic-energy operator $\sqrt{\vec{p}^2 + m^2}$ yields

$$\left\langle \sqrt{\vec{p}^2 + m^2} \right\rangle \leq \sqrt{\langle \vec{p}^2 \rangle + m^2} \quad .$$

By employing this inequality, we obtain for the expectation value $\langle H \rangle$ of the semi-relativistic Hamiltonian H, Eq. (1),

$$\langle H \rangle \;=\; 2\left\langle \sqrt{\vec{p}^2 + m^2} \right\rangle + \langle V \rangle \leq 2\sqrt{\langle \vec{p}^2 \rangle + m^2} + \langle V \rangle$$

$$= 2\frac{\langle \vec{p}^2 \rangle + m^2}{\sqrt{\langle \vec{p}^2 \rangle + m^2}} + \langle V \rangle = \left\langle 2\frac{\vec{p}^2 + m^2}{\sqrt{\langle \vec{p}^2 \rangle + m^2}} + V \right\rangle \quad . \qquad (2)$$

From now on we specify the Hilbert-space vectors in all expectation values to be the eigenstates of our Hamiltonian H. In this case the expectation value of H, $\langle H \rangle$, as appearing, e. g., in (2), becomes the corresponding semi-relativistic energy eigenvalue E, i. e., $E \equiv \langle H \rangle$, and the inequality (2) tells us that this energy eigenvalue is bounded from above by

$$E \leq \left\langle 2\frac{\vec{p}^2 + m^2}{\sqrt{\langle \vec{p}^2 \rangle + m^2}} + V \right\rangle \quad .$$

The operator within brackets on the right-hand side of this inequality is an "effectively semi-relativistic" Hamiltonian H_{eff} which possesses, quite formally, the structure of a nonrelativistic Hamiltonian,

$$H_{\text{eff}} \equiv 2\frac{\vec{p}^2 + m^2}{\sqrt{\langle \vec{p}^2 \rangle + m^2}} + V = 2\,\hat{m} + \frac{\vec{p}^2}{\hat{m}} + V_{\text{eff}} \quad , \qquad (3)$$

but involves, however, the effective mass

$$\hat{m} = \frac{1}{2}\sqrt{\langle \vec{p}^2 \rangle + m^2} \qquad (4)$$

and the effective nonrelativistic potential

$$V_{\text{eff}} = \frac{2\,m^2}{\sqrt{\langle \vec{p}^2 \rangle + m^2}} - \sqrt{\langle \vec{p}^2 \rangle + m^2} + V = 2\,\hat{m} - \frac{\langle \vec{p}^2 \rangle}{\hat{m}} + V \quad . \qquad (5)$$

The effective mass $\hat{m}$ as given by Eq. (4) as well as the constant, i. e., coordinate-independent, term in the effective potential V_{eff} of Eq. (5), $2\,\hat{m} - \langle \vec{p}^2 \rangle/\hat{m}$, obviously depend on the expectation value of the square of the momentum $\vec{p}$, $\langle \vec{p}^2 \rangle$, and will therefore differ for different energy eigenstates.

Motivated by our above considerations, we propose to approximate the true energy eigenvalues E of the semi-relativistic Hamiltonian H of Eq. (1) by the corresponding "effective" energy eigenvalues E_{eff}, defined as the expectation values of some effective Hamiltonian $\tilde{H}_{\text{eff}}$ taken with respect to the eigenstates $|\rangle_{\text{eff}}$ of its own,

$$E_{\text{eff}} = \langle \tilde{H}_{\text{eff}} \rangle_{\text{eff}} \quad ,$$

where the effective Hamiltonian $\tilde{H}_{\text{eff}}$, as far as its form is concerned, is given by Eqs. (3) to (5) but is implicitly understood to involve the

expectation values of $\vec{p}^2$ with respect to the effective eigenstates $|\rangle_{\text{eff}}$ (that is, $\langle \vec{p}^2 \rangle_{\text{eff}}$ in place of $\langle \vec{p}^2 \rangle$):

$$\tilde{H}_{\text{eff}} = 4\,\tilde{m} + \frac{\vec{p}^2 - \langle \vec{p}^2 \rangle_{\text{eff}}}{\tilde{m}} + V \quad,$$

with

$$\tilde{m} = \frac{1}{2}\sqrt{\langle \vec{p}^2 \rangle_{\text{eff}} + m^2} \quad.$$

Accordingly, the effective energy eigenvalues E_{eff} are given by a rather simple formal expression, viz., by

$$E_{\text{eff}} = 4\,\tilde{m} + \langle V \rangle_{\text{eff}} \quad. \tag{6}$$

3 General Strategy of Evaluation

We intend to elaborate our general prescription for the construction of effectively semi-relativistic Hamiltonians $\tilde{H}_{\text{eff}}$ in more detail for the particular case of power-law potentials depending only on the radial coordinate $r \equiv |\vec{x}|$, i. e., for potentials of the form $V(r) = a\,r^n$ with some constant a. The reason for this restriction is twofold:

1. On the one hand, for power-law potentials the virial theorem [5, 6] in its nonrelativistic form [3, 4] appropriate for the present case,

$$\left\langle \frac{\vec{p}^2}{\tilde{m}} \right\rangle_{\text{eff}} = \frac{1}{2} \left\langle r\,\frac{dV(r)}{dr} \right\rangle_{\text{eff}} \quad,$$

   enables us to replace the expectation value of the potential in (6) by a well-defined function of the expectation value of the squared momentum:

$$a\,\langle r^n \rangle_{\text{eff}} = \frac{2}{n}\,\frac{\langle \vec{p}^2 \rangle_{\text{eff}}}{\tilde{m}} \quad.$$

   This implies for the effective energy eigenvalues

$$E_{\text{eff}} = 4\,\tilde{m} + \frac{2}{n}\,\frac{\langle \vec{p}^2 \rangle_{\text{eff}}}{\tilde{m}} \quad. \tag{7}$$

2. On the other hand, we may take advantage of the fact that for power-law potentials it is possible to pass, without change of the fundamental commutation relations between coordinate variables and their canonically conjugated momenta, from the dimensional phase-space variables employed at present to new, dimensionless

phase-space variables and to rewrite the Hamiltonian in form of a Hamiltonian which involves only these dimensionless phase-space variables [3]. The eigenvalues ϵ of this dimensionless Hamiltonian are, of course, also dimensionless [3]. Applying this procedure, we find for the effective energy eigenvalues

$$E_{\text{eff}} - 4\,\tilde{m} + \frac{\langle \vec{p}^2 \rangle_{\text{eff}}}{\tilde{m}} = \left\langle \frac{\vec{p}^2}{\tilde{m}} + a\,r^n \right\rangle_{\text{eff}} = \left(\frac{a^2}{\tilde{m}^n} \right)^{\frac{1}{2+n}} \epsilon \quad .$$

Combining both of the above expressions for E_{eff}, we obtain a relation which allows us to determine $\langle \vec{p}^2 \rangle_{\text{eff}}$ unambiguously in terms of the dimensionless energy eigenvalues ϵ:

$$\langle \vec{p}^2 \rangle_{\text{eff}}^{2+n} = \frac{1}{4} \left(\frac{n}{2+n} \right)^{2+n} a^2\,\epsilon^{2+n}\,(\langle \vec{p}^2 \rangle_{\text{eff}} + m^2) \quad . \tag{8}$$

For a given power n this equation may be solved for $\langle \vec{p}^2 \rangle_{\text{eff}}$. Insertion of the resulting expression into Eq. (7) then yields the corresponding eigenvalue E_{eff} of the effectively semi-relativistic Hamiltonian $\tilde{H}_{\text{eff}}$.

4 Applications

We would like to investigate the capabilities of the effective treatment proposed in the previous sections by discussing some of its implications for some familiar prototypes of interaction potentials, namely, for the harmonic-oscillator, Coulomb, linear, and funnel potential. To this end we compare for the lowest-lying energy eigenstates (which we will label according to the usual spectroscopic notation) the energy eigenvalues E_{eff} resulting from our effective description with the respective energy eigenvalues E_{NR} obtained within the corresponding and by now rather standard nonrelativistic approach [3, 4]. Although for obvious reasons we do not intend at this place to perform a fit of some experimentally observed particle spectrum, we employ, for the purpose of comparison, a set of numerical values of the involved parameters which represents the typical orders of magnitude of the various constants found within a phenomenological description of hadrons as bound states of quarks by (nonrelativistic) potential models [3, 4]. In particular, our choice for the mass m of the bound-state constituents is $m = 1.8$ GeV, which corresponds to the typical mass of the constituent c quark.

Harmonic-oscillator and Coulomb potential may be investigated on purely algebraic grounds. More sophisticated potentials, however, have to be handled with the help of numerical methods [7]. Occasionally, it will prove to be favourable to inspect in particular the ultrarelativistic limit of the developed formalism, defined by vanishing mass m of the bound-state constituents, i. e., by $m = 0$.

An important feature of the experimentally measured mass spectra of hadrons—which may serve to provide a decisive criterion regarding the usefulness of our effective treatment for a meaningful description of hadrons—is the empirically well-established linearity of the Regge trajectories: both mesons and baryons may be grouped to form sets of particles which populate (approximately) linear Regge trajectories; the different members of these sets are related by the fact that, apart from a constant shift, the squares of their masses, i. e., of the energy eigenvalues of the corresponding bound states of quarks in their center-of-momentum frame, are proportional to the relative orbital angular momentum ℓ of the bound-state constituents or, equivalently, the spin of the composite particles, with almost one and the same constant of proportionality, the so-called Regge slope $\beta \simeq 1.2$ GeV^2, for all Regge trajectories [8]. We indicate these relationships by $E^2(\ell) = \beta\,\ell + \text{const.}$ In general, the theoretical dependence of the energy eigenvalues E on the angular momentum ℓ will turn out to be described by some rather complicated function of ℓ. For this reason we only take a quick glance on the asymptotic behaviour of the predicted energy eigenvalues $E(\ell)$ for large values of the angular momentum ℓ, symbolically denoted by the limit $\ell \to \infty$. There we may expect to observe a simple power-law rise of the calculated squares of energy eigenvalues $E^2(\ell)$ for increasing values of ℓ.

4.1   Harmonic oscillator

For the harmonic-oscillator potential $V(r) = a\,r^2$, that is, for $n = 2$, Eq. (8) reduces to a quartic equation for the expectation value $\langle \vec{p}^{\,2} \rangle_{\text{eff}}$. Inserting the well-known expression [3] for the dimensionless energy eigenvalues ϵ of the three-dimensional harmonic oscillator, $\epsilon = 2\,N$, where N is given in terms of the radial and orbital angular-momentum quantum numbers n_r and ℓ, respectively, by $N = 2\,n_r + \ell + \frac{3}{2}$, and

introducing the shorthand notation $k \equiv \sqrt{\frac{a}{2}}\, N$ and

$$x \equiv \left(\frac{k^8}{2}\right)^{\frac{1}{3}} \left[\left(1 + \sqrt{\frac{256}{27}\frac{m^6}{k^4} + 1}\right)^{\frac{1}{3}} + \left(1 - \sqrt{\frac{256}{27}\frac{m^6}{k^4} + 1}\right)^{\frac{1}{3}}\right] \quad,$$

the analytic solution of this quartic equation for $\langle \vec{p}^{\,2}\rangle_{\mathrm{eff}}$ reads

$$\langle \vec{p}^{\,2}\rangle_{\mathrm{eff}} = \frac{\sqrt{x}}{2} + \sqrt{\frac{k^4}{2\sqrt{x}} - \frac{x}{4}} \quad.$$

According to our above prescription, the effective energy eigenvalue is then given by inserting this result into Eq. (7). In the ultrarelativistic limit this effective energy eigenvalue takes a particularly simple form: from $m = 0$ one finds $\langle \vec{p}^{\,2}\rangle_{\mathrm{eff}} = k^{\frac{4}{3}}$ and $E_{\mathrm{eff}} = 4\sqrt{\langle \vec{p}^{\,2}\rangle_{\mathrm{eff}}} = 2\,(4\,a)^{\frac{1}{3}} N^{\frac{2}{3}}$.

Table 1 compares the nonrelativistic and effectively semi-relativistic approaches for the harmonic oscillator. Our choice of parameter values implies for the ground state $E_{\mathrm{NR}} < E_{\mathrm{eff}}$ whereas for the excited states $E_{\mathrm{NR}} > E_{\mathrm{eff}}$ holds.

Table 1: Energy eigenvalues (in GeV) of nonrelativistic and effectively semi-relativistic Hamiltonian for the harmonic-oscillator potential $V(r) = a\,r^2$, with $a = 0.5$ GeV^3.

State	Nonrelativistic Hamiltonian	Effective Hamiltonian
1S	5.181	5.198
1P	6.235	6.188
2S/1D	7.289	7.128

Furthermore, it is no problem to determine immediately the large-ℓ behaviour of the theoretical energy eigenvalues. In the ultrarelativistic case, because of $N \propto \ell$ for large ℓ, the effective energy eigenvalues E_{eff} behave, according to their above-mentioned explicit general form, like $E_{\mathrm{eff}}^2(\ell) \propto \ell^{\frac{4}{3}}$. In contrast to that, the large-ℓ asymptotic behaviour of the corresponding nonrelativistic energy eigenvalues E_{NR} is given by $E_{\mathrm{NR}} = 2\sqrt{\frac{a}{m}}\,\ell + \mathrm{const.}$ [3, 4], which implies $E_{\mathrm{NR}}^2(\ell) \propto \ell^2$. We conclude that for the harmonic-oscillator potential (at least the ultrarelativistic limit of) the effective treatment comes closer to the observed linearity of the Regge trajectories than the nonrelativistic approach.

4.2 Coulomb potential

For the Coulomb potential $V(r) = -\kappa/r$, that is, for $n = -1$, Eq. (8) reduces to a linear equation for the expectation value $\langle \vec{p}^2 \rangle_{\text{eff}}$. Inserting the well-known expression [3] for the dimensionless energy eigenvalues ϵ of the Coulomb problem, $\epsilon = -(2\,N)^{-2}$, where N is given in terms of the radial and orbital angular-momentum quantum numbers n_r and ℓ, respectively, by $N = n_r + \ell + 1$, we obtain from this linear equation for $\langle \vec{p}^2 \rangle_{\text{eff}}$

$$\langle \vec{p}^2 \rangle_{\text{eff}} = \frac{\kappa^2\,m^2}{16\,N^2 - \kappa^2}\quad,$$

and, after inserting this expression into Eq. (7), for the effective energy eigenvalue

$$E_{\text{eff}} = \frac{m}{N}\,\frac{8\,N^2 - \kappa^2}{\sqrt{16\,N^2 - \kappa^2}}\quad.$$

For the case of vanishing orbital angular momentum, i. e., for $\ell = 0$, an analytic expression for the energy eigenvalues E_{SR} of the genuine semi-relativistic Hamiltonian (1) may be derived, which reads [9]

$$E_{\text{SR}} = \frac{2\,m}{\sqrt{1 + \dfrac{\kappa^2}{4\,n^2}}}\quad,\qquad n = 1, 2, \ldots\quad.$$

Within both of the above approaches all energy eigenvalues vanish in the ultrarelativistic limit $m = 0$. For the Coulomb problem, because of the lack of any sort of dimensional parameter inherent to the theory in the case $m = 0$, this kind of degeneracy must take place already for dimensional reasons. It may be understood completely by application of the general, that is, relativistic, virial theorem [5, 6] derived by two of the present authors.

Table 2 compares the nonrelativistic and effectively semi-relativistic approaches for the Coulomb potential. For the numerical values of our parameters employed at present all energy levels calculated within the effective treatment surmount their counterparts in the nonrelativistic description: $E_{\text{eff}} > E_{\text{NR}}$.

Picking up the question of the large-ℓ behaviour of the theoretical energy eigenvalues again, we find from the reported explicit expression that in the limit $\ell \to \infty$ the effective energy eigenvalues E_{eff} will not depend on the orbital angular momentum ℓ at all: $E_{\text{eff}}^2(\ell) \propto \ell^0$. In the nonrelativistic case, on the other hand, the energy eigenvalues E_{NR}

Table 2: Energy eigenvalues (in GeV) of nonrelativistic and effectively semi-relativistic Hamiltonian for the Coulomb potential $V(r) = -\kappa/r$, with $\kappa = 0.456$.

State	Nonrelativistic Hamiltonian	Effective Hamiltonian
1S	3.5064	3.5294
2S/1P	3.5766	3.5824
3S/2P/1D	3.5896	3.5922

behave asymptotically like $E_{\mathrm{NR}} = -\frac{m\,\kappa^2}{4\,\ell^2} + \mathrm{const.}$ [3, 4]. Because of the negative sign in front of the ℓ-dependent term this yields a rise of the form $E_{\mathrm{NR}}^2(\ell) \propto \ell^{-4}$ for increasing values of ℓ.

4.3   Variational method

In general, it will not be possible to find some analytic expressions for the effective energy eigenvalues E_{eff}. However, in order to obtain an approximation to the spectrum of energy eigenvalues to be expected or to get, at least, some idea of it one may adopt the variational method described in the following.

This standard variational method proceeds along the steps of the following, extremely simple recipe [4, 10]:

1. Choose a suitable set of trial states $|\lambda\rangle$. The members of this set are distinguished by some sort of variational parameter λ.

2. Compute the set of expectation values of the Hamiltonian under consideration, H, with respect to these trial states $|\lambda\rangle$ in order to obtain $E(\lambda) \equiv \langle\lambda|H|\lambda\rangle$.

3. Determine that value of the variational parameter λ—say, λ_{min}—which minimizes the resulting, λ-dependent expression $E(\lambda)$.

4. Compute $E(\lambda)$ at the point of the minimum λ_{min} to find in this way the minimal expectation value $E(\lambda_{\mathrm{min}})$ of the Hamiltonian H in the Hilbert-space subsector of the chosen trial states $|\lambda\rangle$.

This minimum $E(\lambda_{\min})$ provides, of course, only an upper bound to the proper energy eigenvalue of the Hamiltonian H.[1]

Application of this straightforward variational procedure to one of our effectively semi-relativistic Hamiltonians $\tilde{H}_{\text{eff}}$ leads to $E_{\text{eff}}(\lambda_{\min})$, which, according to its derivation, represents at least an upper bound to the corresponding effective energy eigenvalue E_{eff}.

Note that, as far as the above variational procedure is concerned, the expectation value $\langle \vec{p}^2 \rangle_{\text{eff}}$ entering in the effective Hamiltonian has to be regarded as a constant. Consequently, it has not to be taken into account in the course of minimization of the energy expression $E(\lambda)$ by varying the characteristic parameter λ. Rather, in the framework of this variational technique, it has to be equated to the expectation value of $\vec{p}^2$ taken with respect to precisely that trial state $|\lambda_{\min}\rangle$ which is characterized by just the minimizing value $\lambda_{\min}$ of the variational parameter λ, that is, to $\langle \lambda_{\min} | \vec{p}^2 | \lambda_{\min} \rangle$.

For the present investigation we adopt the simplest conceivable set of trial states $|\lambda\rangle$, namely, the ones the coordinate-space representation $\psi(\vec{x})$ of which is given, for vanishing radial quantum number n_r, by the Gaussian trial functions (w. l. o. g., $\lambda > 0$)

$$\psi_{\ell m}(r, \theta, \phi) = \sqrt{\frac{2\,\lambda^{2\ell+3}}{\Gamma\left(\ell + \frac{3}{2}\right)}}\, r^\ell \, \exp\left(-\frac{\lambda^2\, r^2}{2}\right) \mathcal{Y}_{\ell m}(\theta, \phi) \quad ,$$

where $\mathcal{Y}_{\ell m}$ denote the spherical harmonics for angular momentum ℓ and projection m, and the normalization factor of these trial functions makes use of the so-called gamma function [12]

$$\Gamma(z) \equiv \int\limits_0^\infty dt\, t^{z-1} \, \exp(-t) \quad .$$

For this particular set of trial functions we obtain for the expectation values of the square $\vec{p}^2$ of the momentum $\vec{p}$ and of the n-th power r^n of the radial coordinate r, respectively, with respect to the trial states $|\lambda\rangle$

$$\langle \lambda | \vec{p}^2 | \lambda \rangle = \left(\ell + \tfrac{3}{2}\right) \lambda^2$$

and

$$\langle \lambda | r^n | \lambda \rangle = \frac{\Gamma\left(\ell + \frac{3+n}{2}\right)}{\Gamma\left(\ell + \frac{3}{2}\right)} \frac{1}{\lambda^n} \quad .$$

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[1] The accuracy of this method is discussed in Ref. [11].

4.4 Linear potential

For the linear potential $V(r) = a\,r$, that is, for $n = 1$, Eq. (8) reduces to a cubic equation for the expectation value $\langle \vec{p}^2 \rangle_{\text{eff}}$ which, of course, may be solved analytically. Unfortunately, for the linear potential the dimensionless energy eigenvalues ϵ are only known [3] for the case of vanishing orbital angular momentum ℓ, i. e., only for $\ell = 0$. In this case they are given by the negative zeros of the Airy function [12]. In any case, that is, for arbitrary values of ℓ, the effective energy eigenvalues E_{eff} may be found by employing some numerical procedure.

However, before performing a numerical computation of the energy eigenvalues E_{eff} of the effective Hamiltonian $\tilde{H}_{\text{eff}}$ with linear potential, we apply the simple variational technique introduced in the preceding subsection. For this Hamiltonian the value of the variational parameter λ which minimizes the expectation value $\langle \lambda | \tilde{H}_{\text{eff}} | \lambda \rangle$, that is, λ_{min}, is implicitly given by

$$\lambda_{\text{min}}^3 = \frac{a}{2} \frac{\Gamma(\ell + 2)}{\Gamma\left(\ell + \frac{5}{2}\right)} \tilde{m} \quad .$$

Recalling the definition of $\tilde{m}$ as given in Sect. 2, we obtain from this expression a cubic equation for $\langle \lambda_{\text{min}} | \vec{p}^2 | \lambda_{\text{min}} \rangle$,

$$\langle \lambda_{\text{min}} | \vec{p}^2 | \lambda_{\text{min}} \rangle^3 = \frac{a^2}{16} \left(\ell + \tfrac{3}{2}\right) \left(\frac{\Gamma(\ell + 2)}{\Gamma\left(\ell + \frac{3}{2}\right)}\right)^2 \left(\langle \lambda_{\text{min}} | \vec{p}^2 | \lambda_{\text{min}} \rangle + m^2\right) \quad ,$$

the analytic solution of which may be written down quickly. Insertion of this result into Eq. (7) yields $E_{\text{eff}}(\lambda_{\text{min}})$ for the linear potential. In the ultrarelativistic limit $m = 0$ we find in this way the (variational) effective energy eigenvalues

$$E_{\text{eff}}(\lambda_{\text{min}}) = 3 \left(\ell + \tfrac{3}{2}\right)^{\frac{1}{4}} \sqrt{\frac{\Gamma(\ell + 2)}{\Gamma\left(\ell + \frac{3}{2}\right)}} \, a \quad .$$

Table 3 compares the nonrelativistic and effectively semi-relativistic approaches for the linear potential. According to our announcement, for the effective treatment the energy eigenvalues have been computed in an iterative way with the help of the numerical procedure developed in Ref. [7]. Again our choice of parameter values places all effective energy levels above their nonrelativistic counterparts: $E_{\text{eff}} > E_{\text{NR}}$. The variational calculation outlined above yields, just for comparison, for the 1S and 1P state, respectively, the (variational) effective energy eigenvalues

$E_{\text{eff}}(\lambda_{\text{min}}) = 4.3107$ GeV and $E_{\text{eff}}(\lambda_{\text{min}}) = 4.6168$ GeV, both of which are very close to the numerically obtained results and, of course, upper bounds to them.

Table 3: Energy eigenvalues (in GeV) of nonrelativistic and effectively semi-relativistic Hamiltonian for the linear potential $V(r) = a\,r$, with $a = 0.211$ GeV^2.

State	Nonrelativistic Hamiltonian	Effective Hamiltonian
1S	4.2812	4.3087
1P	4.5793	4.6149
2S	4.7911	4.8309

Inspecting once again the large-ℓ behaviour of the predicted energy eigenvalues, we may read off from the above explicit expression for the ultrarelativistic (variational) effective energy eigenvalues $E_{\text{eff}}(\lambda_{\text{min}})$, with the help of a useful relation describing the asymptotic behaviour of the ratio of gamma functions [12], viz.,

$$\lim_{\ell \to \infty} \frac{\Gamma(\ell + z)}{\Gamma(\ell + u)} = \ell^{z-u} \quad ,$$

the very pleasing result $E_{\text{eff}}^2(\lambda_{\text{min}}) = 9\,a\,\ell$. Accordingly, the effectively semi-relativistic energy eigenvalues of the linear potential are perfectly able to reproduce the observed linearity of the Regge trajectories with, however, a slope which is slightly larger than the one obtained within different analyses based on the true semi-relativistic Hamiltonian (1), all of which end up with one and the same finding: $E_{\text{SR}}^2 = 8\,a\,\ell$ [13, 14]. Moreover, from the point of view of a correct description of the linear Regge trajectories, both of the semi-relativistic treatments are clearly superior to the corresponding nonrelativistic approach, which gives for the energy eigenvalues of the linear potential [3, 4]

$$E_{\text{NR}} = 3 \left(\frac{a^2}{4\,m} \right)^{\frac{1}{3}} \ell^{\frac{2}{3}} + \text{const.}$$

and therefore $E_{\text{NR}}^2(\ell) \propto \ell^{\frac{4}{3}}$.

4.5 Funnel potential

Unfortunately, the potentials considered up to now are merely of more or less academic interest. Finally, however, we would like to discuss a potential which has been among the first ones to be proposed [17] for the description of hadrons as bound states of constituent quarks, viz., the funnel (or Cornell or Coulomb-plus-linear) potential.

This funnel potential comprehends the two basic ingredients of any realistic, that is, phenomenologically acceptable, inter-quark potential, namely,

- at short inter-quark distances some Coulomb-like singularity of perturbative origin, which arises from one-gluon exchange, and

- at large inter-quark distances an approximately linear rise of non-perturbative origin, which is responsible for colour confinement.

The funnel potential incorporates these two features in the simplest conceivable manner:
$$V(r) = -\frac{\kappa}{r} + a\,r \quad .$$
In this form it still represents the prototype of almost all forthcoming potential models designed to describe all the (binding) forces acting between quarks.[2]

This funnel potential is, beyond doubt, not of the power-law type. Consequently, it cannot be subjected to the general effective formalism developed so far but deserves a special treatment, which might consist of some purely numerical approach.

However, as before, we first want to obtain some insight by applying the variational procedure described in Subsect. 4.3. The value λ_{min} of the variational parameter λ which minimizes for the case of the above funnel potential the expectation value of the effective Hamiltonian $\tilde{H}_{\mathrm{eff}}$ with respect to our Gaussian trial states is (because of the presence of $\tilde{m}$ only implicitly) determined by the relation
$$\lambda_{\mathrm{min}}^3 = \frac{\tilde{m}}{2}\frac{\Gamma(\ell+1)}{\Gamma\left(\ell+\frac{5}{2}\right)}\left[\kappa\,\lambda_{\mathrm{min}}^2 + a\,(\ell+1)\right] \quad .$$

In the ultrarelativistic limit $m = 0$ this relation fixes λ_{min} to
$$\lambda_{\mathrm{min}} = \sqrt{\frac{a\,\Gamma(\ell+2)}{4\sqrt{\ell+\frac{3}{2}}\,\Gamma\left(\ell+\frac{3}{2}\right) - \kappa\,\Gamma(\ell+1)}} \quad ,$$

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[2] For a brief survey see, for instance, Ref. [3].

which, in turn, implies for the (variational) effective energy eigenvalues of the funnel potential

$$E_{\mathrm{eff}}(\lambda_{\min}) = 2\,\lambda_{\min}\left(3\sqrt{\ell + \tfrac{3}{2}} - \kappa\,\frac{\Gamma(\ell + 1)}{\Gamma\left(\ell + \tfrac{3}{2}\right)}\right) \quad .$$

The large-ℓ behaviour of the ultrarelativistic (variational) effective energy eigenvalues $E_{\mathrm{eff}}(\lambda_{\min})$ resulting from this expression is the same as for the pure linear potential: $E_{\mathrm{eff}}^2(\lambda_{\min}) = 9\,a\,\ell$. This circumstance is an unavoidable consequence of the fact that in the limit $\ell \to \infty$ all contributions of the Coulomb part of the funnel potential to the above effective energy eigenvalues vanish, which may be seen immediately by recalling once more the above-mentioned asymptotic behaviour of the ratio of gamma functions. Accordingly, for very large orbital angular momenta ℓ the positioning of the energy levels of the funnel potential is controlled by its confinement part only.[3]

Table 4 compares the nonrelativistic and effectively semi-relativistic approaches, as before, and, in addition, also the truly semi-relativistic treatment for the funnel potential. The results for both nonrelativistic and effectively semi-relativistic approach have been computed by just the same numerical procedure [7] as before whereas the results for the semi-relativistic treatment have been obtained, also only on numerical footing, by the so-called "method of orthogonal collocation" [18]. This method tries to approximate the action of the square-root operator of the relativistically correct kinetic energy, $\sqrt{\vec{p}^2 + m^2}$, on some suitably chosen (truncated) set of basis states by a well-defined (finite) matrix representation. For our choice of parameter values, in particular, for a mass m of the bound-state constituents as large as $m = 1.8$ GeV, we find that the effective approach is worse than the nonrelativistic one: $E_{\mathrm{SR}} < E_{\mathrm{NR}} < E_{\mathrm{eff}}$ for all states.

Table 5, however, by diminishing step by step the numerical value of the mass m of the bound-state constituents while retaining at the same time the numerical values of the potential parameters κ and a, demonstrates by the example of the obtained ground-state energy level that this situation may change drastically when reducing the value of m. Below $m \simeq 0.75$ GeV we find (for the 1S state) $E_{\mathrm{SR}} < E_{\mathrm{eff}} < E_{\mathrm{NR}}$. These inequalities indicate that for comparatively small masses m the

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[3]A similar observation has already been made in Ref. [14] within a slightly different context [15, 16].

Table 4: Energy eigenvalues (in GeV) of nonrelativistic, effectively semi-relativistic, and truly semi-relativistic Hamiltonian for the funnel potential $V(r) = -\kappa/r + a\,r$, with $\kappa = 0.456$ and $a = 0.211$ GeV^2.

State	Nonrelativistic Hamiltonian	Effective Hamiltonian	Semi-relativistic Hamiltonian
1S	3.9655	4.0172	3.9250
1P	4.4014	4.4471	4.3707
2S	4.5873	4.6378	4.5217

effective treatment represents a better approximation to the true, i. e., semi-relativistic, energy eigenvalues than the nonrelativistic approach.

Table 5: Dependence of the ground-state (1S) energy eigenvalues (in GeV) on the mass m of the bound-state constituents within nonrelativistic, effectively semi-relativistic, and truly semi-relativistic approach for the funnel potential. The parameter values used here are the same as in Table 4.

m [GeV]	Nonrelativistic Hamiltonian	Effective Hamiltonian	Semi-relativistic Hamiltonian
0.250	1.6677	1.5712	1.4078
0.336	1.6996	1.6540	1.5066
0.500	1.8539	1.8524	1.7256
0.750	2.1904	2.2149	2.1054
1.000	2.5807	2.6168	2.5161

Table 6, by confronting the relative deviations of the nonrelativistic ($E_{\rm NR}$) and effectively semi-relativistic ($E_{\rm eff}$) energy eigenvalues from their proper semi-relativistic ($E_{\rm SR}$) counterparts (normalized to $E_{\rm SR}$), works out this tendency still more clearly: for masses m of the bound-state constituents restricted (for the present set of parameter values) by $m < 0.75$ GeV the relative errors of our effective treatment become smaller than those of the nonrelativistic one. (A further, rather trivial lesson to be learned from the relative differences of energy eigenvalues compared by Table 6 reads: the heavier the bound-state constituents are, the more irrelevant the precise form of the employed kinematics

becomes and, consequently, the more the results of the different ways of description will approach each other—the energy of the bound state is dominated by the sum of the masses of the constituents.)

Table 6: Relative differences of the energy eigenvalues of the (1S) ground state within nonrelativistic, effectively semi-relativistic, and truly semi-relativistic treatment of the funnel potential. The parameter values used here are the same as in Table 4.

m [GeV]	$\dfrac{E_{\mathrm{NR}} - E_{\mathrm{SR}}}{E_{\mathrm{SR}}}$ [%]	$\dfrac{E_{\mathrm{eff}} - E_{\mathrm{SR}}}{E_{\mathrm{SR}}}$ [%]
0.250	18.5	11.6
0.336	12.8	9.8
0.500	7.4	7.3
0.750	4.0	5.2
1.000	2.6	4.0
1.800	1.5	2.6

5   Conclusions

The present work has been dedicated to the formulation of effectively semi-relativistic Hamiltonians which are designed in such a way that— by suitable interpretation of their (effective) parameters—they allow us to approximate an entirely semi-relativistic formalism at a formally nonrelativistic level. Application of the developed formalism to a few representative static interaction potentials gave indications that below some specific critical mass of the involved particles, where the obtained energy eigenvalues are closer to the (exact) semi-relativistic ones than those of a nonrelativistic description, the effective approach represents, at least in relativistic regions, an improvement of the certainly rather crude nonrelativistic approximation. Simultaneously, this observation might contribute to the eventual explanation of the surprising success of (a variety of) nonrelativistic potential models in describing hadrons as bound states of quarks even for the case of relativistically moving constituents.

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